International Journal of Business Management and Commerce; Vol. 5 No. 1; April 2020
International Journal of Business Management and Commerce
Published by Center for Contemporary Research

Description of Income and Substitution Effects using Slutsky Identity

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Abstract

In the paper we describe how income and price changes affect consumer's decision making using the Slutsky Identity. We also investigate economical meaning of curve turn-movements of consumer's budget. For analyzing the changes of consumer's choice we are using two step movements (turn, then parallel movement) of budget curve.

Key words: Demand, Income Effect, Slutsky equation, Substitution Effect, Budget Curve

Dependence of the costumer's behavior change on the environment is one of the main interests in economics. The question is, how does customer's choice change while changing the price of goods? It is common to think that the demand on the goods is decreasing while prices on goods are increasing. However, optimal demand on goods, called the Giffen Goods, could decrease despite the fall of price on goods. What is the reason of such a strange behavior of consumers? Why can the price change cause such a different effect on demand? Changes in price can cause two main effects. First, changes in norm by which one good can be exchanged by another can occur. Second, gross purchasing ability of income will change as well. The first effect, related to the demand change while changing the goods interchange norm, is called Substitution Effect; and the second effect, related to the demand change caused by increase in purchasing ability, is called Income Effect.

To obtain much more exact definition of those effects we should divide price change into two parts: firstly, consider comparative price change and regulate money income so that purchase ability stays the same; secondly, leave prices unchanged and regulate purchase ability. A good explanation of this is shown in the Figure 1. Here we have a situation, when price of "good-1" has decreased. This means that budget curve will turn around the vertical brake point m / p_2 and it's slope reduces (here, *m* is amount of income and *p* is price). Such movement of budget curve can be divided in two steps: first turn budget curve around the initial basket, then move it in parallel to the right so, it goes through new required basket.

This „turn-shuffle“operation, simplifies the division process of demand in to two parts. First step – turn- is a movement when slope of budget curve is changing and purchases ability stays the same. The second step is a movement, when slope is same and purchase ability is changing. This division is a just hypothetical action – the consumer is choosing just new basket as a response on price changes. But to understand how the consumers choice is changing, it's convenient to present the changes of budget curves by this two-stage movement - first turn, then movement in parallel.

Turning of the curve. Does this budget curve turn-shuffle process have any economical meaning? First of all let's consider the **turned curve. (See Fig.1).** Here we have the same slope and same price ratio as last curve case. But despite that the money incomes related to this curve are different, due to the difference in the vertical brake-down. This consumer basket is achievable –but not optimal because the initial basket (x_1, x_2) is located on the turned-budget curve. The purchase ability of consumer is unchanged from the point of view, that initial consumer basket is just achievable basket for the new curve.

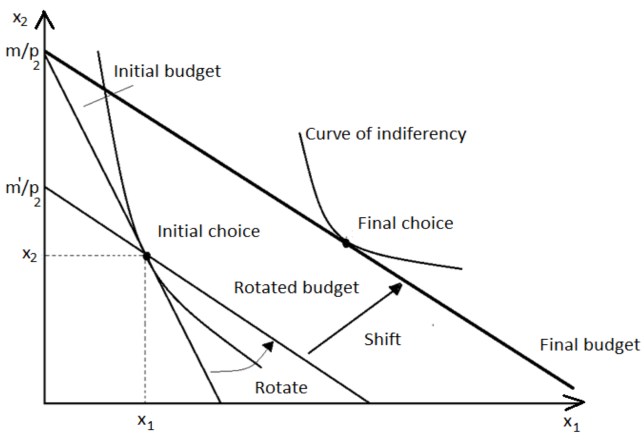

Fig.1. Rotation and shift. **When price on good 1 is changing, and income is constant, the budget curve will rotate around the vertical ex. We are presenting this changes by two steps: first, the budget curve will rotate around the initial basket, then - will shift parallel to the right so it cross the new demanded basket.**

Let us calculate how to regulate money income while the old consumer basket remains the same. Denote money income value with variable *m*, for which the initial consumer basket is just available. This will be money flow related to the turned budget curve. Because (x_1, x_2) is available for both (p_1, p_2, m) and (p_1', p_2, m'), we have:

$$m' = p_1' \, x_1 + p_2 \, x_2$$

$$m = p_1 \, x_1 + p_2 \, x_2$$

If we subtract second equation from the first one we obtain:

$$m' - m = x_1 [\, p_1' - p_1].$$

This equation tells us that changes in money income, which requires for the old consumer basket to remain the same, equals to the product of amount of "good-1" and difference between price changes.

If $\Delta p_1 = p_1' - p_1$ is the price changes of the "good-1", and $\Delta m = m' - m$ is the necessary change of income for old basket to remain available, then:

$$\Delta m = x_1 \Delta p_1 \qquad (1)$$

Note that the signs of the change of income and the price are always the same. For old basket to remain available, the income should increase while price increases. We have a formula for the budget turn curve: this is a new budget curve with a new price and income change Δm . Notice, that when the price of good-1 is falling, the change in income is negative. When the price decreases, the purchase ability of the consumer increases. In this case we have to reduce consumer's income for keeping his/her purchase ability at the same level. Also, when price increases, the purchase ability decreases, thats why the change of income should be positive.

In general, the fact that (x_1, x_2) is achievable, does not mean that it is an optimal basket for the budget turn curve. Figure 2 shows the optimal basket, denoted by Y, related to the budget turn curve. This is an optimal goods basket, where we change the price and then regulate the income to keep old basket achievable. The movement from X to Y is called Substitution Effect. It shows how consumer is „substituting"one good with another while prices change and purchase ability remains the same.

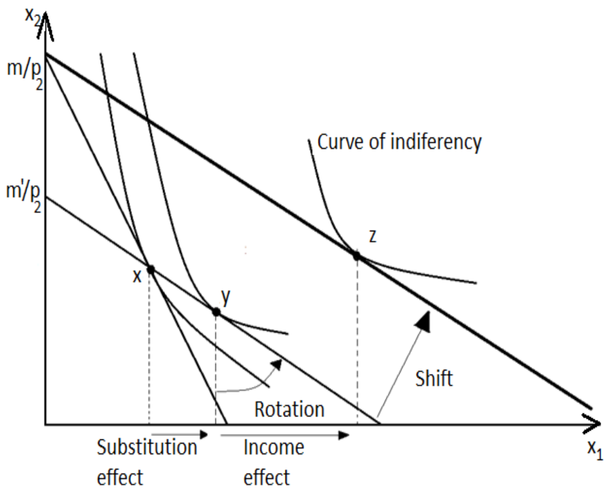

Fig.2. Substitution effect and income effect. **Rotation gives substitution effect, shift – income effect.**

To be more precise, substitution effect Δx_1^s is a change of demand on the good-1, while price of the good-1 becomes p_1' and at the same time money income turns to m':

$$\Delta x_1^s = x_1(p_1', m') - x_1(p_1, m).$$

Moving of the curve. Let`s go back to the second stage of price regulation – **displacement**. The economical interpretation of this action is very easy. We know that budget curve moves to the right when income is changing while the prices remain the same. That's why the second stage of price regulation is called the Income Effect. We only change consumer's income from m' to m, but leave the (p_1, p_2) prices unchanged. This change is shown in Figure 2 by moving from (y_1,y_2) point to (z_1,z_2). It is natural to call this movement income effect because all we are doing is changing the income when leaving the prices unchanged. To be more precise, income effect Δx_1^n represents a change of demand on the good-1, while changing the value of income from m' to m and leaving prices p_1' of good-1 unchanged:

$$\Delta x_1^n = x_1(p_1', m) - x_1(p_1', m')$$

The total change of gross demand Δx_1 is the change related to the price, while income is the same:

$$\Delta x_1 = x_1(p_1', m) - x_1(p_1, m).$$

This change can be presented by two changes – substitution effect and income effect, as we saw above. If we use familiar variables we will obtain:

$$\Delta x_1 = \Delta x_1^s + \Delta x_1^n$$

$$x_1(p_1', m) - x_1(p_1, m) = [x_1(p_1', m') - x_1(p_1, m)] + [x_1(p_1', m) - x_1(p_1', m')].$$

This identity shows that total change in demand is equal to the sum of substitution and income effects and is called the Slutsky Identity. We already saw that effects of substitution and income can be presented graphically as a combination of turning and shifting or now algebraically using Slutsky identity, where everything is represented in terms of absolute changes. More often this identity is presented in terms of the relative changes. When Slutsky identity is presented with relative changes, we find that it is convenient to define Δx_1^m as an opposite number to income effect [1][2]:

$$\Delta x_1^m = x_1(p_1', m') - x_1(p_1', m) = -\Delta x_1^n$$

If use this definition the Slutsky identity will obtain the following form:

$$\Delta x_1 = \Delta x_1^s - \Delta x_1^m .$$

divide both sides by Δp_1

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} - \frac{\Delta x_1^m}{\Delta p_1} . \qquad (2)$$

The first term of right hand side of this equation is relative change in income when the price is variable and income is regulated for the old basket to be achievable, thus it is substitution effect. We will see below that the second term is income effect. From equation (1) we can find Δp_1

$$\Delta p_1 = \frac{\Delta m}{x_1} .$$

Plug this expression in equation (2) to obtain :

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} - \frac{\Delta x_1^m}{\Delta m} x_1 .$$

This is the Slutsky identity expressed in the relative changes. It is possible to interpret all terms in this equation in following manner:

$$\frac{\Delta x_1}{\Delta p_1} = \frac{x_1(p_1', m) - x_1(p_1, m)}{\Delta p_1}$$

is the change in demand while price is changing and income remains the same.

$$\frac{\Delta x_1^s}{\Delta p_1} = \frac{x_1(p_1', m') - x_1(p_1', m)}{\Delta p_1}$$

is the changes in demand, while price is changing and income is regulated so the old basket stays achievable, this is a substitution effect.

$$\frac{\Delta x_1^m}{\Delta m} x_1 = \frac{x_1(p_1', m') - x_1(p_1', m)}{m' - m} x_1 \qquad (3)$$

is the change in demand while the price remains the same and income is regulated, this is a substitution effect. The product in income effect itself has two main terms: the income related demand change and initial level of demand. Change in income caused by income effects when price is changing by Δp_1 and is given as:

$$\Delta x_1^m = \frac{x_1(p_1', m') - x_1(p_1', m)}{\Delta m} x_1 \Delta p_1.$$

but $x_1 \Delta p_1$ income change is necessary for old basket to remain achievable, thus $x_1 \Delta p_1 = \Delta m$. The changes in demand suitable to income effect will look like:

$$\Delta x_1^m = \frac{x_1(p_1', m') - x_1(p_1', m)}{\Delta m} \Delta m$$

That's what we had before. The Slutsky identity represents the dependency of the change in demand on the change in price as sum of the substitution and income effects. The income effect is a result of change in price followed by change in gross purchasing ability. But in our case there are two reasons for changes in purchasing ability. The first effect is in the definition of the Slutsky identity. For example, when price is reduced you can purchase the same amount of goods as in the past and save some money. Let's call it the income effect. The second effect is relatively new - when the price of goods is changing it causes the changes in initial basket value and thus changes in money income.

In the previous form of the Slutsky identity, the money income was fixed. Now we should take into account changes in money income while initial basket value changes. Thus, when we calculate effect of price change on demand the Slutsky equation takes the following form:

Whole change of demand= changes caused by substitutional effect+ demand changes caused by common income effect + demand changes caused by initial income effect.

It can be written in the following form:

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} - x_1 \frac{\Delta x_1^m}{\Delta m} + \textit{Initial income effect} \quad (4)$$

How does the last term in this equation look? When initial price of basket is changing, money income is changing as well, and this causes changes in demand. Thus, initial income effect contains two parts:

Initial income effect= demand changes, when income is changing $\times$ *income change, when price is changing* (5)

First look at the second effect. According to the definition, income is

$$m = p_1\omega_1 + p_2\omega_2,$$

and

$$\frac{\Delta m}{\Delta p_1} = \omega_1.$$

This equation shows how money income changes, while price on the "good-1" changes. The first term of equation (5) shows how demand changes when income is changing. We already have an expression that demonstrates this ratio between the changes of the demand and the income $\Delta x_1^m / \Delta m$. Thus, initial income effect will have the following form:

$$Initial\ icome\ effect = \frac{\Delta x_1^m}{\Delta m}\frac{\Delta m}{\Delta p_1} = \frac{\Delta x_1^m}{\Delta m}\omega_1 \qquad (6)$$

Substituting equation (6) in equation (4) we will obtain the final form of the Slutsky identity:

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} + (\omega_1 - x_1)\frac{\Delta x_1^m}{\Delta m}.$$

In the Slutsky equation shown above, we have missed one small inaccuracy. While considering the effects of the changes of the initial market value on the changes in demand, we said that this effect is equal to $\Delta x_1^m/\Delta m$. In the previous version of Slutsky equation this was change in demand while income was changing so, that the initial basket remained achievable. But it is not necessary for this value of the change to be equal to demand change when initial basket value is changing. Let`s consider this in details. Suppose price of good-1 change from p_1 to p_1' . Denote by m''corresponding new money income for p_1' price. As we know from definition of m'':

$$m'' - m = \Delta p_1 \omega_1$$

note that the following equation is also true:

$$\frac{x_1(p_1', m'') - x_1(p_1, m)}{\Delta p_1} =$$

$$+\frac{x_1(p_1', m') - x_1(p_1', m)}{\Delta p_1} \text{ (Substitution effect)}$$

$$-\frac{x_1(p_1', m') - x_1(p_1', m)}{\Delta p_1} \text{ (Common income effect)}$$

$$+\frac{x_1(p_1', m'') - x_1(p_1', m)}{\Delta p_1} \text{ (Initial income effect).}$$

According to the definition of income effect:

$$\Delta p_1 = \frac{m' - m}{x_1},$$

on the other hand, according to the definition of initial income effect:

$$\Delta p_1 = \frac{m'' - m}{\omega_1}.$$

After proper transformations the Slutsky identity obtains the following form:

$$\frac{x_1(p_1', m'') - x_1(p_1, m)}{\Delta p_1} =$$

$$+\frac{x_1(p_1', m) - x_1(p_1, m)}{\Delta p_1} \quad \text{(Substitution effect)}$$

$$-\frac{x_1(p_1', m') - x_1(p_1', m)}{m' - m} x_1 \quad \text{(Common income effect)}$$

$$+\frac{x_1(p_1', m'') - x_1(p_1', m)}{m'' - m} \omega_1 \quad \text{(Initial income effect).}$$

Rewrite this equation using symbol Δ, we will get:

$$\frac{\Delta x_1}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} - \frac{\Delta x_1^m}{\Delta m} x_1 + \frac{\Delta x_1^\omega}{\Delta m} \omega_1 .$$

The last new term is the change in demand on the "good-1" when changing income multiplied by the initial amount of the "good-1". This is indeed the initial income effect. Suppose that we are considering very small changes in the price and corresponding incomes that are small as well. In this case we can consider that fractions in both effects are the same values, because "good-1" changes approximately the same amount when income changes from m to m' and from m to m''. In case of such small changes, we can sum the last two terms of income effects and present it in the following form:

$$\frac{\Delta x_1^m}{\Delta m}(\omega_1 - x_1),$$

That's where the Slutky`s identity shown above comes from:

$$\frac{\Delta x_1^t}{\Delta p_1} = \frac{\Delta x_1^s}{\Delta p_1} + (\omega_1 - x_1)\frac{\Delta x_1^m}{\Delta m}.$$

It is possible to present Slutsky identity using differential by taking the limit of this equation, or calculating it by taking partial differential. Suppose, the demand function of "good-1" is $x_1(p_1, m(p_1))$ when the price of the "good-2" is fixed and income depends on the price of the "good-1" in following form $m(p_1) = p_1\omega_1 + p_2\omega_2$. Then we will have:

$$\frac{dx(p_1, m(p_1))}{dp_1} = \frac{\partial x_1(p_1, m)}{\partial p_1} + \frac{\partial x_1(p_1, m)}{\partial m}\frac{dm(p_1)}{dp_1}.$$

From definition of $m(p_1)$ we know how income is changing during the price changes:

$$\frac{\partial m(p_1)}{\partial p_1} = \omega_1 \qquad (7)$$

and from the Slutsky equation we know how demand is changing due to price change, while the income is fixed.

$$\frac{\partial x_1(p_1,m)}{\partial p_1} = \frac{\partial x_1^s(p_1)}{\partial p_1} - \frac{\partial x(p_1,m)}{\partial m} x_1. \qquad (8)$$

By substituting equation (8) in equation (7) we will obtain:

$$\frac{dx_1(p_1,m(p_1))}{dp_1} = \frac{\partial x_1^s(p_1)}{\partial p_1} + \frac{\partial (p_1,m)}{\partial m}(\omega_1 - x_1),$$

For us this is suitable form of Slutsky equation.

So far we defined substitution effect as the change in demand when the prices are variable, but the purchasing ability remains unchanged. This is one of the definitions of the substitution effect, but there is other definition that describes something called Hicks substitution effect.

Assume that instead of rotating budget curve around the initial consumer basket we translate it along the indifference curve that passes through initial basket, shown in figure 3. In this manner we are offering to the consumers the budget curve that has the same ratio of prices of the final curve, but corresponding incomes are different. According to this new curve, the purchasing ability is not enough to buy the basket that is identical to the initial basket. In addition, we need to note that each definition of the substitution effect has its place and each of them can be used depending on the given problem. It can be shown that for a small variation of price, these two definitions of the effect are identical.

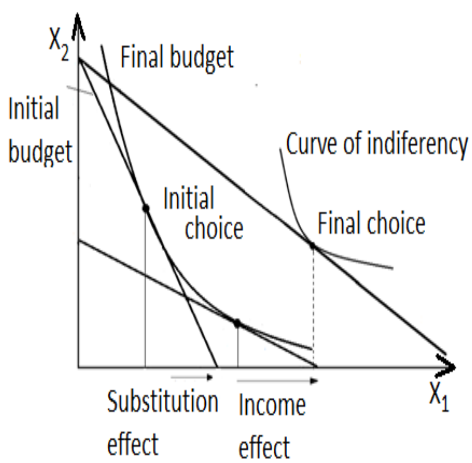

Fig.3. Hicks Substitution effect. Here, the budget curve does not rotate around the initial choice, but moves toward the indifference curve.

The demand is changing while prices are changing in three different cases: in case of fixed income (standard case), in case of fixed purchasing ability (Slutsky substitution effect) and in case of fixed utility (Hicks substitution effect). It is possible to form the dependency of the price and demand in all three cases; as a result we will get three different curves: standard demand curve, Slutsky demand curve, and Hicks demand curve. All three curves are decaying curves in the case of normal goods (i.e. demand increases with income). As an exception, for inferior goods (i.e. demand declines as the level of income) the curve can be increasing. [3]

Hicks demand curve often is called compensated demand curve. It is important to note that the compensated demand curve is very important in studying income-cost, but this requires heavier mathematical calculations than given in this article and is beyond our goal. [4]

As a next step it would be interesting to investigate what effect the change of price on "goods-2" will have on substitution effect, using the Slutsky identity. Hicks method is related to more logical theoretical principles, whereas Slutsky method gives more statistical solution of quantitative problem.

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